BANACH GABOR FRAMES WITH HERMITE FUNCTIONS: POLYANALYTIC SPACES FROM THE HEISENBERG GROUP

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Dedicated to Paul L. Butzer on the occasion of his 80th birthday

Abstract. Gabor frames with Hermite functions are equivalent to sampling sequences in true Fock spaces of polyanalytic functions. In the L^2-case, such an equivalence follows from the unitarity of the polyanalytic Bargmann transform. We will introduce Banach spaces of polyanalytic functions and investigate the mapping properties of the polyanalytic Bargmann transform on modulation spaces. By applying the theory of coorbit spaces and localized frames to the Fock representation of the Heisenberg group, we derive explicit polyanalytic sampling theorems which can be seen as a polyanalytic version of the lattice sampling theorem discussed by J. M. Whittaker in Chapter 5 of his book *Interpolatory Function Theory*.

1. Introduction

In this note we will be concerned with *Gabor expansions* of the form

$$f(t)=\sum_{l,k\in\mathbb{Z}}c_{k,l}e^{2\pi i\alpha lt}h_n(t-\beta k), \tag{1.1}$$

where α and β are real constants and h_n are the *Hermite functions*

$$h_n(t)=c_ne^{\pi t^2}\left(\frac{d}{dt}\right)^n\left(e^{-2\pi t^2}\right)$$

where c_n is chosen so that $\|h_n\|_2=1$. Expansions of type (1.1) are useful for the multiplexing of signals [4], [28], [1] and image processing [22].

Gabor expansions with Hermite functions have been introduced in mathematical time-frequency analysis [27] and studied further in [19], [28] and [1], with an emphasis on vector-valued Gabor frames (so-called Gabor super-frames).

The properties of Gabor expansions with Hermite functions are the result of an interplay between classical analysis (orthogonal polynomials, Weierstrass sigma functions) and modern mathematical methods (frame theory, group representations, and modulation spaces).

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It has been discovered recently [1] that the construction of expansions of type (1.1) is equivalent to sampling problems in spaces of functions which satisfy the generalized Cauchy-Riemann equations of the form

$$\left(\frac{d}{d\overline{z}}\right)^n F(z) = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial \xi}\right)^n F(x+i\xi) = 0. \tag{1.2}$$

This is equivalent to saying that $F(z)$ is a polynomial of order $n-1$ in $\overline{z}$ with analytic functions $\{\varphi_k(z)\}_{k=0}^{n-1}$ as coefficients:

$$F(z) = \sum_{k=0}^{n-1} \overline{z}^k \varphi_k(z) \tag{1.3}$$

Such functions are known as *polyanalytic functions.* They have been investigated thoroughly, notably by the Russian school led by Balk [5], and they provide extensions of classical operators from complex analysis [6]. The connection of polyanalytic function theory to Gabor expansions seems to be yet another instance of how — in the words of Folland [17] — "the abstruse meets the applicable" in time-frequency analysis. Indeed, time-frequency analysis is prone to reveal unexpected relations to other fields of mathematics. Two recent examples are the associations with Banach algebras [25] and with noncommutative geometry [21].

Our objective is to investigate the connection between Gabor frames and polyanalytic function theory in more detail. Our main contributions are the following:

(1) We will obtain explicit formulas for the sampling and interpolation of polyanalytic functions on a lattice in the complex plane.
(2) We extend the theory of the polyanalytic Fock space from Hilbert space to Banach spaces and study polyanalytic functions satisfying L^p-condition. Precisely, we investigate the space $\mathcal{F}_p^n(\mathbb{C})$ consisting of all polyanalytic functions F of order n, i.e., satisfying (1.2), such that
$$\|F\|^p = \int_{\mathbb{C}} |F(z)|^p \, e^{-\pi p \frac{|z|}{2}^2} dz$$
is finite. For this purpose we use the theory of modulation spaces [10] and develop the L^p-theory of the polyanalytic Bargmann transform which so far has only been studied in the L^2 case [1].
(3) We then extend the frame and sampling expansions beyond the Hilbert space. The tools are taken from coorbit theory [14], [13], [23] and the theory of localized frames [26].

As a specific result we state a polyanalytic version of a sampling formula for a lattice. The sampling theorem is in the spirit of the fundamental Whittaker-Shannon-Kotel´nikov formula, as discussed for instance in Whittaker's classical treatise *Interpolatory Function Theory* [34]. Let $\Lambda = \alpha(\mathbb{Z}+i\mathbb{Z})$ be the square lattice consisting of the points $\lambda = \alpha l + i\alpha m$, $k, m \in \mathbb{Z}$ and let $\sigma_{\Lambda^\circ}(z)$ be the classical Weierstrass sigma function associated to the adjoint lattice $\Lambda^\circ = \alpha^{-1}(\mathbb{Z}+i\mathbb{Z})$ and

set

$$S^n_{\Lambda^0}(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} \frac{(\sigma_{\Lambda^0}(z))^{n+1}}{n!z}\right].$$

Theorem. *If $\alpha^2 < \frac{1}{n+1}$, then every $F \in \mathcal{F}_p^{n+1}(\mathbb{C})$ possesses the sampling expansion*

$$F(z) = \sum_{\lambda \in \alpha(\mathbb{Z}+i\mathbb{Z})} F(\lambda) e^{\pi\overline{\lambda}z - \pi|\lambda|^2/2} S^n_{\Lambda^0}(z-\lambda),$$

with unconditional convergence in $\mathcal{F}_p^n(\mathbb{C})$ for $1 \le p < \infty$.

The outline of the paper is as follows. In section 2, we explain the connection between the short-time Fourier transformwith Hermite functions and the true polyanalytic Bargmann transform. We extend the L^2-theory of the true polyanalytic Bargmann transform to a general class of Banach spaces of polyanalytic functions in section 3. Then section 4 studies Gabor frames with Hermite functions in $L^2(\mathbb{R})$. We provide a different proof of the sufficient condition given in [27] for the lattice parameters which generate those frames. In the last section we study Gabor Banach frames with Hermite functions using coorbit theory and localized frames associated with the representations of the Heisenberg group, and derive the corresponding sampling theorems for the polyanalytic Fock spaces.

2. Gabor transforms with Hermite functions

2.1. **The Bargmann transform.** Expansions of the type (1.1) are closely related to the samples of the short-time Fourier transform of f with respect to the Hermite windows h_n.

We recall that the short-time Fourier transform of a function or distribution f with respect to a window function g is defined to be

$$V_g f(x,\xi) = \int_{\mathbb{R}} f(t)\overline{g(t-x)} e^{-2\pi i \xi t} dt. \tag{2.1}$$

If we choose the Gaussian function $h_0(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$ as a window in (2.1), then a simple calculation shows that

$$e^{-i\pi x\xi + \pi\frac{|z|^2}{2}} V_{h_0} f(x,-\xi) = \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi z^2 - \frac{\pi}{2}t^2} dt = \mathcal{B}f(z). \tag{2.2}$$

Here $\mathcal{B}f(z)$ is the usual Bargmann transform of f, see for instance [16]. The Bargmann transform $\mathcal{B}f$ is an entire function and thus satisfies the Cauchy-Riemann equation

$$\frac{d}{d\overline{z}} \mathcal{B}f = 0.$$

Furthermore, $\mathcal{B}$ is an unitary isomorphism from $L^2(\mathbb{R})$ onto the Bargmann-Fock space $\mathcal{F}(\mathbb{C})$ consisting of all entire functions satisfying

$$\|F\|^2_{\mathcal{F}(\mathbb{C})} = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dz < \infty. \tag{2.3}$$

2.2. **The polyanalytic Bargmann transform.** Now choose a general Hermite function h_n as a window for the short-time Fourier transform in (2.1). A calculation (see [8] or [27] for details) shows that

(2.4)
$$e^{-i\pi x\xi+\frac{\pi}{2}|z|^2}V_{h_n}f(x,-\xi)=\left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}}\sum_{0\leq k\leq n}\binom{n}{k}(-\pi\overline{z})^k\left(\frac{d}{dz}\right)^{n-k}(\mathcal{B}f)(z)=F(z).$$

Now we have a serious obstruction for exploiting complex analysis tools. The function F on the right hand side of (2.4) is no longer analytic. However, after differentiating (2.4) $n+1$ times with respect to $\overline{z}$, we see that F satisfies the equation

$$\left(\frac{d}{d\overline{z}}\right)^{n+1}F(z)=0. \tag{2.5}$$

A function F satisfying (2.5) is called polyanalytic of order $n+1$. By using the Leibnitz formula, we may write (2.4) more compactly as

$$e^{-i\pi x\xi+\frac{\pi}{2}|z|^2}V_{h_n}f(x,-\xi)=\left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}}e^{\pi|z|^2}\frac{d^n}{dz^n}\left[e^{-\pi|z|^2}\mathcal{B}f(z)\right]. \tag{2.6}$$

This motivates the following definition:

$$\mathcal{B}^{n+1}f(z)=e^{-i\pi x\xi+\frac{\pi}{2}|z|^2}V_{h_n}f(x,-\xi). \tag{2.7}$$

The map $\mathcal{B}^n$ is called the *true polyanalytic Bargmann transform* of order n and has been studied in [1] and [2]. As a consequence of the orthogonality relations for the short-time Fourier transform it was shown that $\|\mathcal{B}^n f\|_{\mathcal{F}(\mathbb{C})}=\|f\|_{L^2(\mathbb{R})}$. See also [20] for an alternative approach. Furthermore, the range $\mathcal{B}^n(L^2(\mathbb{R}))$ under $\mathcal{B}^n$ consists exactly of all polyanalytic functions F satisfying the integrability condition $\int_{\mathbb{C}}|F(z)|^2e^{-\pi|z|^2}\,dz<\infty$ and such that, for some entire function H,

$$F(z)=\left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}}e^{-\pi|z|^2}\left(\frac{d}{dz}\right)^n\left[e^{-\pi|z|^2}H(z)\right].$$

We denote the range of $\mathcal{B}^n$ by $\mathcal{F}^n(\mathbb{C})=\mathcal{B}(L^2(\mathbb{R}))$ and call $\mathcal{F}^n(\mathbb{C})$ the true polyanalytic Fock space of order n, see [1], [2] and [35]. The prefix "true" has been used by Vasilevski [35] to distinguish them from the polyanalytic Fock space $\mathbf{F}^n(\mathbb{C})$, which consists of *all* polyanalytic functions up to order n. This space possesses the orthogonal decomposition:

$$\mathbf{F}^n(\mathbb{C})=\mathcal{F}^1(\mathbb{C})\oplus...\oplus\mathcal{F}^n(\mathbb{C}), \tag{2.8}$$

We remark that the orthogonality of the $\mathcal{F}^k(\mathbb{C})$ follows directly from the orthogonality relations of the short-time Fourier transform and (2.7). The spaces $\mathcal{F}^n(\mathbb{C})$ also appear in connection to the eigenspaces of an operator related to the Landau levels [3].

The orthogonal decomposition (2.8) suggests a second transform. Define $\mathbf{B}^n:L^2(\mathbb{R},\mathbb{C}^n)\to\mathbf{F}^n(\mathbb{C})$ by mapping each vector $\mathbf{f}=(f_1,...,f_n)\in L^2(\mathbb{R},\mathbb{C}^n)$ to the

following polyanalytic function of order n:

$$\mathbf{B}^n\mathbf{f} = \mathcal{B}^1 f_1 + ... + \mathcal{B}^n f_n \,. \tag{2.9}$$

This map is again a Hilbert space isomorphism and is called the *polyanalytic Bargmann transform* [1].

The polyanalytic Bargmann transform possesses an interesting interpretation in signal processing.

(1) Given n signals $f_1, \ldots, f_n$, with finite energy ($f_k \in L^2(\mathbb{R})$ for every k), process each individual signal by evaluating $\mathcal{B}^k f_k$. This encodes each signal into one of the n orthogonal spaces $\mathcal{F}^1(\mathbb{C}), \ldots, \mathcal{F}^n(\mathbb{C})$.
(2) Construct a new signal $F = \mathbf{Bf} = \mathcal{B}^1 f_1 + ... + \mathcal{B}^n f_n$ as a superposition of the n processed signals.
(3) Sample, transmit, or process F.
(4) Let P^k denote the orthogonal projection from $\mathbf{F}^n(\mathbb{C})$ onto $\mathcal{F}^k(\mathbb{C})$, then $P^k(F) = \mathcal{B}^k f_k$ by virtue of (2.8).
(5) Finally, after inverting each of the transforms $\mathcal{B}^k$, we recover each component f_k in its original form.

The combination of n independent signals into a single signal $\mathbf{B}^n\mathbf{f}$ and the subsequent processing are referred to as multiplexing.

The projection P^k can be written explicitly as an integral operator, see Proposition 3 and [2].

3. L^p theory of polyanalytic Fock spaces

3.1. **The spaces $\mathbf{F}_p^n$ and $\mathcal{F}_p^n$.** We next introduce the L^p version of the polyanalytic Bargmann-Fock spaces. For $p \in [1, \infty[$ write $\mathcal{L}_p(\mathbb{C})$ to denote the Banach space of all measurable functions equiped with the norm

$$\|F\|_{\mathcal{L}_p(\mathbb{C})} = \left(\int_{\mathbb{C}} |F(z)|^p \, e^{-\pi p \frac{|z|^2}{2}} \, dz \right)^{1/p}.$$

For $p = \infty$, we have $\|F\|_{\mathcal{L}_\infty(\mathbb{C})} = \sup_{z \in \mathbb{C}} |F(z)| \, e^{-\pi \frac{|z|^2}{2}}$. With this notation we extend the definitions of polyanalytic Fock spaces to the Banach space setting.

Definition 1. *We say that a function F belongs to the* polyanalytic Fock space $\mathbf{F}_p^n(\mathbb{C})$*, if $\|F\|_{\mathcal{L}_p(\mathbb{C})} < \infty$ and F is polyanalytic of order n.*

Definition 2. *We say that a function F belongs to the* true polyanalytic Fock space $\mathcal{F}_p^{n+1}(\mathbb{C})$ *if $\|F\|_{\mathcal{L}_p(\mathbb{C})} < \infty$ and there exists an entire function H such that*

$$F(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{-\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} H(z) \right].$$

Clearly, $\mathcal{F}_p^1(\mathbb{C}) = \mathcal{F}_p(\mathbb{C})$ is the standard Bargmann-Fock space. The space $\mathcal{F}_1^1(\mathbb{C})$ is the complex version of the Feichtinger algebra [11, 12], and it will play an important role in last section of the paper.

3.2. **Orthogonal decompositions.** In dealing with polyanalytic functions the following version of integration by parts is useful. The disc of radius r is denoted by $\mathbf{D}_r$ as usual, its boundary is $\delta\mathbf{D}_r$.

Lemma 1. *If $f, g \in C^1(\mathbf{D}_r)$, then*

$$\int_{\mathbf{D}_r} f(z) \frac{d}{d\overline{z}} \overline{g(z)} dz = -\int_{\mathbf{D}_r} \frac{d}{d\overline{z}} f(z) \overline{g(z)} dz + \frac{1}{i} \int_{\delta\mathbf{D}_r} f(z) \overline{g(z)} dz, \tag{3.1}$$

where the line integral over the circle $\delta\mathbf{D}_r$ is oriented counterclockwise.

Iterating (3.1) one obtains the formula

$$\begin{aligned} \int_{\mathbf{D}_r} f(z) \left(\frac{d}{d\overline{z}}\right)^n \overline{g(z)} dz &= (-1)^n \int_{\mathbf{D}_r} \left(\frac{d}{d\overline{z}}\right)^n f(z) \overline{g(z)} dz \\ &+ \frac{1}{i} \sum_{j=0}^{n-1} (-1)^j \int_{\delta\mathbf{D}_r} \left(\frac{d}{d\overline{z}}\right)^j f(z) \left(\frac{d}{d\overline{z}}\right)^{n-j-1} \overline{g(z)} dz. \end{aligned} \tag{3.2}$$

The Lemma follows from Green´s formula. It can also be seen directly for polyanalytic polynomials $p(z, \overline{z})$ and then extended by density.

Now recall that the monomials

$$e_n(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n$$

provide an orthonormal basis for $\mathcal{F}_2(\mathbb{C})$ and that $\mathcal{B}(h_n)$=e_n. In addition, they are orthogonal on every disk D_r: for every $r > 0$,

$$\int_{D_r} e_n(z) \overline{e_m(z)} e^{-\pi \frac{|z|^2}{2}} dz = C(r, m)\, \delta_{nm}. \tag{3.3}$$

The normalization constant $C(r, m)$ depends on r and m and satisfies $\lim_{r\to\infty} C(r, m) = 1$. The double orthogonality follows easily by writing the integral in polar coordinates (see [24, pg. 54]). It will be the key fact behind the proof of the next result.

Proposition 1. *The sequence of polyanalytic orthogonal polynomials defined as*

$$e_{k,m}(z) = e^{\pi|z|^2} \left(\frac{d}{dz}\right)^k \left[e^{-\pi|z|^2} e_m(z)\right],$$

enjoy the following properties:

1. *The linear span of $\{e_{k,m} : m \geq 0, 0 \leq k \leq n\}$ is dense in $\mathbf{F}_p^{n+1}(\mathbb{C})$ for $1 \leq p < \infty$ (and weak-* dense in $\mathbf{F}_\infty^{n+1}(\mathbb{C})$.*
2. *For fixed n the linear span of $\{e_{n,m} : m \geq 0\}$ is dense in $\mathcal{F}_p^{n+1}(\mathbb{C})$ for $1 \leq p < \infty$.*
3. *$\mathcal{B}^k(h_m)$=$\left(\frac{\pi^m}{m!}\right)^{\frac{1}{2}} e_{k,m}$.*

Proof. To prove completeness of $\{e_{k,m}\}$ in $\mathbf{F}_p^n(\mathbb{C})$, suppose that $F \in \mathbf{F}_p^n(\mathbb{C})$ is such that $\langle F, e_{k,m}\rangle_{\mathcal{L}_p(\mathbb{C})} = 0$, for all $0 \leq k \leq n-1$ and $m \geq 0$. For $n = 1$, $k = 0$, the

classical argument for the completeness of the monomials shows that $F=0$. For $k\geq 1$, we use formula (3.2) as follows:

$$\int_{\mathbf{D}_r} F(z)\overline{e_{k,m}(z)}e^{-\pi|z|^2}dz = \int_{\mathbf{D}_r} F(z)e^{\pi|z|^2}\left(\frac{d}{d\overline{z}}\right)^k\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]e^{-\pi|z|^2}dz$$

$$\begin{aligned}
&= \int_{\mathbf{D}_r} F(z)\left(\frac{d}{d\overline{z}}\right)^k\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz \\
&= (-1)^k\int_{\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^k F(z)\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz \\
&\quad +\frac{1}{i}\sum_{j=0}^{n-1}(-1)^j\int_{\delta\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^j F(z)\left(\frac{d}{d\overline{z}}\right)^{k-j-1}\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz.
\end{aligned}$$

Now, using the representation (1.3) we can write the polyanalytic function F in the form:

$$F(z)=\sum_{0\leq p\leq n-1}\overline{z}^p\sum_{l\geq 0}c_{l,p}z^l.$$

Since the sum converges uniformly over compact set, we interchange the order of summation and integration in the following manipulations.

$$\begin{aligned}
&\int_{\delta\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^j F(z)\left(\frac{d}{d\overline{z}}\right)^{k-j-1}\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz \\
&= \int_{\delta\mathbf{D}_r}\sum_{j\leq p\leq n-1}p...(p-j+1)\overline{z}^{p-j}\sum_{l\geq 0}c_{l,p}z^l\left(\frac{d}{d\overline{z}}\right)^{k-j-1}\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz \\
&= \sum_{j\leq p\leq n-1}p...(p-j+1)\overline{z}^{p-j}\sum_{l\geq 0}c_{l,p}\int_{\delta\mathbf{D}_r}\overline{z}^{p-k}z^j\left(\frac{d}{d\overline{z}}\right)^{k-j-1}\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz
\end{aligned}$$

If we let $r\to\infty$, the integral on the last expression approaches zero and since the function $\varphi_p(z)=\sum_{l\geq 0}c_{l,p}z^l$ is analytic, the coefficients $\{c_{l,p}\}_{l\geq 0}$ decay fast enough in order to assure that the whole expression approaches zero. Thus,

$$\begin{aligned}
\int_{\mathbf{D}_r} F(z)\overline{e_{k,m}(z)}e^{-\pi|z|^2}dz &= (-1)^k\int_{\mathbf{D}_r}\left(\frac{d}{d\overline{z}}\right)^k F(z)\left[e^{-\pi|z|^2}\overline{e_m(z)}\right]dz \\
&= (-1)^k\sum_{k\leq p\leq n}\sum_{j\geq 0}c_{j,p}\frac{p...(p-k+1)\pi^{m/2}}{\sqrt{m!}}\int_{\mathbf{D}_r}z^j\overline{z}^{m+p-k}e^{-\pi|z|^2}dz.
\end{aligned}$$

We first use (3.3) and then let $r\to\infty$. The hypothesis $0=\langle F,e_{k,m}\rangle_{\mathcal{F}(\mathbb{C}^d)}$ for $0\leq k\leq n$ implies that

$$\sum_{k\leq p\leq n}\frac{p...(p-k+1)(p+m-k)!}{\pi^{\frac{3}{2}+p-k}\sqrt{m!}}c_{m+p-k,p}=0,\quad m\geq 0,\ 0\leq k\leq n.$$

Solving the resulting triangular system for each m, we obtain $c_{j,p}=0$ for $k\leq p\leq n-1$ and $j\geq 0$. Therefore $F=0$.

To prove item (2), suppose now that $F \in \mathcal{F}_p^{n+1}(\mathbb{C})$. Then there exists an entire function $H(z) = \sum_{j\geq 0} a_j z^j$ such that

$$F(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{-\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} H(z)\right] = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} \sum_{0\leq k\leq n} \binom{n}{k} (-\pi\overline{z})^k \left(\frac{d}{dz}\right)^{n-k} H(z).$$

We apply Lemma (1) and denote by $B(r)$ the boundary terms arising from (3.2). Then

$$\begin{aligned}
\int_{\mathbf{D}_r} F(z)\overline{e_{n,m}(z)}e^{-\pi|z|^2}dz &= \int_{\mathbf{D}_r} F(z) \left(\frac{d}{d\overline{z}}\right)^n \left[e^{-\pi|z|^2}\overline{e_m(z)}\right] dz + B(r) \\
&= (-1)^n \int_{\mathbf{D}_r} \left(\frac{d}{d\overline{z}}\right)^n F(z)\overline{e_m(z)}e^{-\pi|z|^2}dz + B(r) \\
&= (-1)^n \int_{\mathbf{D}_r} H(z)\overline{e_m(z)}e^{-\pi|z|^2}dz + B(r) \\
&= \sum_{j\geq 0} a_j \int_{\mathbf{D}_r} z^j\overline{e_m(z)}e^{-\pi|z|^2}dz + B(r) \\
&= C(r,m)a_m + B(r)\,,
\end{aligned}$$

where we have used (3.3) in the last equality. If $r \to \infty$, then $B(r) \to 0$, and the hypothesis $0 = \langle F, e_{n,m}\rangle_{\mathcal{F}(\mathbb{C})}\,, m \geq 0$ now implies that $a_j = 0, j \geq 0$. Thus $F = 0$.

Assertion (3) of the proposition is an immediate consequence of the definition of the true polyanalytic Bargmann transform and the fact that $\mathcal{B}(h_m){=}e_m$ (see also [1]). ■

An obvious consequence of the above proposition is the extension of the orthogonal decomposition (2.8) to the p-norm setting. Similar results appear in [31] for the unit disk case. See also [7] for other approaches to decompositions in spaces of polyanalytic functions.

Corollary 1. *The following decompositions hold for* $1 < p < \infty$*:*

$$\begin{aligned}
\mathbf{F}_p^n(\mathbb{C}) &= \mathcal{F}_p^1(\mathbb{C}) \oplus ... \oplus \mathcal{F}_p^n(\mathbb{C}). \\
\mathcal{L}_p(\mathbb{C}) &= \bigoplus_{n=1}^{\infty} \mathcal{F}_p^n(\mathbb{C}).
\end{aligned}$$

3.3. **Mapping properties of the true polyanalytic Bargmann transform in modulation spaces.**

For the investigation of the mapping properties of the true polyanalytic Bargmann transform $\mathcal{F}_p^n(\mathbb{C})$ we need the concept of *modulation space*. Following [24], the modulation space $M^p(\mathbb{R})$, $1 \leq p \leq \infty$, consists of all tempered distributions f such that $V_{h_0}f \in L^p(\mathbb{R}^2)$ equipped with the norm

$$\|f\|_{M^p(\mathbb{R})} = \|V_{h_0}f\|_{L^p(\mathbb{R}^2)}\,.$$

Modulation spaces are ubiquitous in time-frequency analysis. They were introduced by Feichtinger in [10].

With a view to studying sampling sequences in poly-Fock spaces $\mathcal{F}_p^n(\mathbb{C})$ for general p, we prove some statements concerning the properties of the true poly-Bargmann transform, which may be of independent interest.

Proposition 2. *There exist constants C, D, such that, for every $f \in M^p(\mathbb{R})$, $1 \leq p \leq \infty$,*

$$C \left\| \mathcal{B}^n f \right\|_{\mathcal{L}_p(\mathbb{C})} \leq \left\| \mathcal{B} f \right\|_{\mathcal{L}_p(\mathbb{C})} \leq D \left\| \mathcal{B}^n f \right\|_{\mathcal{L}_p(\mathbb{C})} . \tag{3.4}$$

Proof. This follows from the theory of modulation spaces: since the definition of Modulation space is independent of the particular window chosen [24, Proposition 11.3.1], then the norms

$$\|f\|'_{M^p(\mathbb{R}^2)} = \|V_{h_n} f\|_{L^p(\mathbb{R}^2)}$$

and

$$\|f\|_{M^p(\mathbb{R}^2)} = \|V_{h_0} f\|_{L^p(\mathbb{R}^2)} ,$$

are equivalent. Therefore, there exist constants C, D, such that

$$C \left\| V_{h_n} f \right\|_{L^p(\mathbb{R}^2)} \leq \left\| V_{h_0} f \right\|_{\mathcal{F}_p(\mathbb{C})} \leq D \left\| V_{h_n} f \right\|_{L^p(\mathbb{R}^2)} .$$

By definition of $\mathcal{B}^n$ and $\mathcal{B}$, this yields (3.4). ■

The next result includes the surjectivity of the transform $\mathcal{B}^n$ onto $\mathcal{F}_p^n(\mathbb{C})$.

Corollary 2. *Given $F \in \mathcal{F}_p^n(\mathbb{C})$ there exists $f \in M^p(\mathbb{R})$ such that $F = \mathcal{B}^n f$. Moreover, there exist constants C, D such that:*

$$C \left\| F \right\|_{\mathcal{L}_p(\mathbb{C})} \leq \left\| \mathcal{B} f \right\|_{\mathcal{L}_p(\mathbb{C})} \leq D \left\| F \right\|_{\mathcal{L}_p(\mathbb{C})} . \tag{3.5}$$

Proof. Since the Hermite functions belong to M^p and $\mathcal{B}^n(h_k) = e_{k,n}$, the range of $\mathcal{B}^n$ contains a set which is dense in $\mathcal{F}_p^n(\mathbb{C})$ for $1 \leq p < \infty$. Thus, $\mathcal{B}^n : L^2(\mathbb{R}) \to \mathcal{F}_p^n(\mathbb{C})$ is onto for $1 \leq p < \infty$. Then (3.4) is equivalent to (3.5).

For $p = \infty$ we use that the span of the $e_{k,m}$ is weak-* dense in $\mathcal{F}_\infty^n(\mathbb{C})$. ■

3.4. **The polyanalytic projection.** Let

$$K^n(w, z) = \frac{1}{n!} e^{\pi |w|^2} \left(\frac{d}{dw} \right)^n \left[e^{\pi \overline{z} w - \pi |w|^2} (w - z)^n \right]$$

denote the reproducing kernel of $\mathcal{F}_2^{n+1}(\mathbb{C})$ and define the integral operator P^n by

$$(P^n F)(w) = \int_{\mathbb{C}} F(z) K^n(w, z) e^{-\pi |z|^2} dz.$$

We have shown in [2] that P^n is the orthogonal projection from $\mathcal{L}_2(\mathbb{C})$ onto the true polyanalytic Fock space $\mathcal{F}_2^{n+1}$. We now show that P^n is also bounded on $\mathcal{L}_p(\mathbb{C})$ and extend the reproducing property to $\mathcal{F}_p^{n+1}$.

Proposition 3. *The operator P^n is bounded from $\mathcal{L}_p(\mathbb{C})$ to $\mathcal{F}_p^{n+1}$ for $1 \leq p \leq \infty$. Moreover, if $F \in \mathcal{F}_p^{n+1}$ then $P^n F = F$.*

Proof. The kernel K^n is

$$K^n(w,z) = \frac{1}{n!} e^{\pi |w|^2} \left(\frac{d}{dw}\right)^n \left[e^{\pi \bar{z} w - \pi |w|^2} (w-z)^n \right] = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-\pi |w-z|^2)^k e^{\pi \bar{z} w} \,,$$

so

$$P^n F(w) e^{-\pi |w|^2/2} = \int_{\mathbb{C}} F(z) e^{-\pi |z|^2/2} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-\pi |w-z|^2)^k e^{\pi \bar{z} w} \right) e^{-\pi |w|^2/2}\, e^{-\pi |z|^2/2}\, dz \,.$$

We take absolute values and observe that $e^{-\pi |w-z|^2/2} = |e^{\pi \bar{z} w}| e^{-\pi |w|^2/2} e^{-\pi |z|^2/2}$, in this way we obtain that

$$|P^n F(w) e^{-\pi |w|^2/2}| \leq \int_{\mathbb{C}} |F(z)| e^{-\pi |z|^2/2} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-\pi |w-z|^2)^k \right) e^{-\pi |w-z|^2/2}\, dz \,.$$

Now set $\Phi(z) = |F(z)| e^{-\pi |z|^2/2} \in L^p(\mathbb{R}^2)$ and $\Psi(z) = \left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-\pi |w-z|^2)^k \right) e^{-\pi |w-z|^2/2} \in L^1(\mathbb{R}^2)$. Then

$$\begin{aligned} \left(\int_{\mathbb{C}} |P^n F(w)|^p e^{-\pi p |w|^2/2}\, dw \right)^{1/p} &\leq \|\Phi * \Psi\|_p \\ &\leq \|\Phi\|_p \, \|\Psi\|_1 \\ &= C \left(\int_{\mathbb{C}} |F(z)|^p e^{-\pi p |z|^2/2}\, dz \right)^{1/p} = \|F\|_{\mathcal{L}_p} \,, \end{aligned} \tag{3.6}$$

and thus P^n is bounded on $\mathcal{L}_p$.

Next set $H(w) = \int_{\mathbb{C}} F(z)(w-z)^n e^{\pi \bar{z} w}\, e^{-\pi |z|^2}\, dz$. Since $F \in \mathcal{L}_p$, the integral is well-defined and H is an entire function. Since

$$P^n F(w) = \frac{1}{n!} e^{\pi |w|^2} \left(\frac{d}{dw}\right)^n \left(e^{-\pi |w|^2} \int_{\mathbb{C}} F(z)(w-z)^n e^{\pi \bar{z} w}\, e^{-\pi |z|^2}\, dz \right) ,$$

it follows that $P^n F$ is a true polyanalytic function. By the boundedness of P^n, we also have $P^n F \in \mathcal{F}_p^{n+1}$. Finally, if $F \in \mathcal{F}_2^{n+1} \cap \mathcal{F}_p^{n+1}$, then $P^n F = F$ by the reproducing kernel property in $\mathcal{F}_2^{n+1}$. Since $\mathcal{F}_2^{n+1} \cap \mathcal{F}_p^{n+1}$ is dense in $\mathcal{F}_p^{n+1}$ by Lemma 1, the identity $P^n F = F$ extends to all $F \in \mathcal{F}_p^{n+1}$. ■

4. Gabor frames in L^2

Stable Gabor expansions of the form (1.1) can be obtained from frame theory. Given a point $\lambda = (\lambda_1, \lambda_2)$ in phase-space $\mathbb{R}^2$, the corresponding time-frequency shift is

$$\pi_\lambda f(t) = e^{2\pi i \lambda_2 t} f(t - \lambda_1), \quad t \in \mathbb{R}.$$

Using this notation, the short-time Fourier transform of a function f with respect to the window g can be written as

$$V_g f(\lambda) = \langle f, \pi_\lambda g \rangle_{L^2(\mathbb{R})} \,.$$

In analogy to the time-frequency shifts π_λ, we use the Bargmann-Fock shifts β_λ defined for functions on $\mathbb{C}$ by

$$\beta_\lambda F(z) = e^{\pi i \lambda_1 \lambda_2} e^{\pi \bar{\lambda} z} F(z-\lambda)\, e^{-\pi|\lambda|^2/2}\,.$$

We observe that the true polyanalytic Bargmann transform intertwines the time-frequency shift π_λ and the Bargmann-Fock representation β_λ on $\mathcal{F}^n$ by a calculation similar to [24, p. 185]:

$$\mathcal{B}^n\left(\pi_\lambda \gamma\right)(z) = \beta_\lambda \mathcal{B}^n \gamma(z)\,, \tag{4.1}$$

for $\gamma \in L^2(\mathbb{R})$. For a countable subset $\Lambda \in \mathbb{R}^2$, one says that the Gabor system $\mathcal{G}(h_n, \Lambda) = \{\pi_\lambda h_n : \lambda \in \Lambda\}$ is a *Gabor frame* or *Weyl-Heisenberg frame* in $L^2(\mathbb{R})$, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A \|f\|^2_{L^2(\mathbb{R})} \leq \sum_{\lambda\in\Lambda} \left| \langle f, \pi_\lambda h_n \rangle_{L^2(\mathbb{R})} \right|^2 \leq B \|f\|^2_{L^2(\mathbb{R})}\,. \tag{4.2}$$

4.1. **A polyanalytic interpolation formula for $\mathcal{F}^n(\mathbb{C})$.** Consider the lattice $\Lambda = \{m_1\lambda_1 + m_2\lambda_2; m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{C}$ spanned by the periods $\lambda_1, \lambda_2 \in \mathbb{C}$, where $\operatorname{Im}(\lambda_1/\lambda_2) > 0$. The size of Λ is the area of the parallelogram spanned by λ_1 and λ_2 in $\mathbb{C}$. If we identify $\mathbb{R}^2$ and $\mathbb{C}$, then we can write Λ as $\Lambda = A\mathbb{Z}^2$ where $A = [\lambda_1, \lambda_2]$ is an invertible real 2×2 matrix. Then the size of the lattice is $s(\Lambda) = |\det A|$.

Let σ be the Weierstrass sigma function corresponding to Λ defined by

$$\sigma(z) = z \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}\,.$$

It is then possible to choose an exponent $a = a(\Lambda) \in \mathbb{C}$, such that the modified sigma function $\sigma_\Lambda(z)$ associated to the lattice Λ

$$\sigma_\Lambda(z) = \sigma(z) e^{a(\Lambda) z^2}$$

satisfies the growth estimate

$$|\sigma_\Lambda(z)| \lesssim e^{\frac{\pi}{2s(\Lambda)}|z|^2}. \tag{4.3}$$

See for instance Proposition 3.5 in [28]. We will only work with the modified sigma function σ_Λ.

Our discussion of sampling theorems for polyanalytic functions will be based on the following function related to the Weierstrass sigma function:

$$S^n_\Lambda(z) = e^{\pi|z|^2} \left(\frac{d}{dz}\right)^n \left[e^{-\pi|z|^2} \frac{\sigma_\Lambda(z)^{n+1}}{n!\, z} \right] \tag{4.4}$$

Then by definition S^n_Λ is polyanalytic of order $n+1$.

Before stating our result, recall that the set Λ is *an interpolating sequence* for $\mathcal{F}^n(\mathbb{C})$ if, for every sequence $\{a_\lambda\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)$, there exists $F \in \mathcal{F}^n(\mathbb{C})$ such that

$$F(\lambda)\, e^{-\frac{\pi}{2}|\lambda|^2} = a_\lambda,$$

for every $\lambda \in \Lambda$.

By means of S^n_Λ we can now formulate an explicit solution to the interpolation problem on Λ for $\mathcal{F}^n$.

Theorem 1. *If $s(\Lambda) > n+1$, then Λ is an interpolating sequence for $\mathcal{F}^{n+1}(\mathbb{C})$. Moreover, the interpolation problem is solved by*

$$F(z) = \sum_{\lambda \in \Lambda} a_\lambda e^{\pi \overline{\lambda} z - \pi|\lambda|^2/2} \, S_\Lambda^n(z-\lambda), \tag{4.5}$$

Proof. The growth estimate (4.3) implies that

$$\left| \frac{\sigma_\Lambda^{n+1}(z)}{z} \right| \lesssim e^{\frac{\pi(n+1)}{2s(\Lambda)}|z|^2}.$$

Since $s(\Lambda) > n+1$, we have $\sigma_\Lambda(z)^{n+1}/z \in \mathcal{F}_2(\mathbb{C})$. By unitarity of the Bargmann transform, there exists a $\gamma = \gamma_{n,\Lambda} = \gamma_\Lambda \in L^2(\mathbb{R})$ such that $\mathcal{B}\gamma_\Lambda(z) = \sigma_\Lambda(z)^{n+1}/z$. Furthermore, since $|V_{h_0}\gamma_\Lambda(z)| = |\mathcal{B}\gamma_\Lambda(z)| \, e^{-\pi|z|^2/2}$, it follows that $\gamma_\Lambda \in M^1(\mathbb{R})$ (or even in the Schwartz class).

Comparing (2.6) and (4.4) we find that

$$S_\Lambda^n(z) = \left(\frac{\pi^n}{n!} \right)^{\frac{1}{2}} (\mathcal{B}^{n+1}\gamma_\Lambda)(z),$$

and $S_\Lambda^n \in \mathcal{F}_2^{n+1}(\mathbb{C})$. As in the proof of [28, Thm. 1.1] we show that S_Λ^n is interpolating on Λ. Using the Leibniz formula, we expand S_Λ^n as

$$S_\Lambda^n(z) = \sum_{k=0}^{n} \binom{n}{k} (-\pi \bar{z})^k \left(\frac{d}{dz} \right)^{n-k} \left(\frac{\sigma_\Lambda^{n+1}(z)}{n! \, z} \right)$$

Since $\sigma_\Lambda^{n+1}(z)/z$ has zeros of order $n+1$ at $\lambda \in \Lambda \setminus \{0\}$ and a zero of order n at $\lambda = 0$, it follows that $S_\Lambda^n(\lambda) = \delta_{\lambda,0}$ for $\lambda \in \Lambda$. Consequently, if F is defined by (4.5), then $F(\lambda)e^{-\pi|\lambda|^2/2} = a_\lambda$, and F is indeed an interpolation of the sequence (a_λ).

It remains to show that the interpolation series (4.5) converges in $\mathcal{F}^{n+1}$. For this we need the additional information that γ_Λ is in $M^1(\mathbb{R})$ and the intertwining property (4.1). Since $\gamma_\Lambda \in M^1(\mathbb{R})$, the series $\sum_{\lambda \in \Lambda} a_\lambda \pi_\lambda \gamma_\Lambda$ converges unconditionally in $L^2(\mathbb{R})$ (e.g., by [24, Thm. 12.2.4]). Therefore the series

$$\begin{aligned}
F(z) &= \sum_{\lambda \in \Lambda} a_\lambda e^{\pi \overline{\lambda} z - \pi|\lambda|^2/2} \, S_\Lambda^n(z-\lambda) \\
&= \sum_{\lambda \in \Lambda} a_\lambda \beta_\lambda S_\Lambda^n(z) \\
&= \sum_{\lambda \in \Lambda} a_\lambda \beta_\lambda \mathcal{B}^{n+1}\gamma_\Lambda(z) \\
&= \sum_{\lambda \in \Lambda} a_\lambda \mathcal{B}^{n+1}(\pi_\lambda \gamma_\Lambda)(z) \\
&= \mathcal{B}^{n+1}\Big(\sum_{\lambda \in \Lambda} a_\lambda \pi_\lambda \gamma_\Lambda \Big)(z)
\end{aligned}$$

converges in $\mathcal{F}^{n+1}$, and the proof is completed. ∎

4.2. **Gabor frames with Hermite functions on $L^2(\mathbb{R})$.** Following Feichtinger and Kozek [15], the adjoint lattice Λ^0 is defined by the commuting property as

$$\Lambda^0 = \{\mu \in \mathbb{R}^2 : \pi_\lambda \pi_\mu = \pi_\mu \pi_\lambda, \text{ for all } \lambda \in \Lambda\}.$$

If $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, then $\Lambda^0 = \beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$.

There exists a remarkable duality between the Gabor systems with respect to Λ^0 and those with respect to Λ. This is often referred to as the *Janssen-Ron-Shen duality principle* [29, 32].

Theorem A (*Duality principle*). *The Gabor system $\mathcal{G}(g, \Lambda)$ is a* frame *for $L^2(\mathbb{R})$ if and only if the Gabor system $\mathcal{G}(g, \Lambda^0)$ is a Riesz basis for its closed linear span in $L^2(\mathbb{R})$.*

Combining the duality principle with Theorem 1, one recovers the result of [27]:

Theorem 2. *If $s(\Lambda) < \frac{1}{n+1}$, then the Gabor system $\mathcal{G}(h_n, \Lambda)$ is a frame for $L^2(\mathbb{R})$.*

Proof. First observe that $s(\Lambda^0) = \frac{1}{s(\Lambda)}$. If $s(\Lambda) < \frac{1}{n+1}$, then $s(\Lambda^0) > n+1$. It follows from Theorem 1 that the lattice Λ^0 is an interpolating sequence for $\mathcal{F}_2^{n+1}(\mathbb{C})$. Since

$$\langle f, \pi_\lambda h_n \rangle_{L^2(\mathbb{R})} = V_{h_n} f(x, \xi) = e^{i\pi x\xi - \frac{\pi}{2}|z|^2} \mathcal{B}^{n+1} f(z),$$

then it is clear that $\mathcal{G}(h_n, \Lambda^0)$ is a Riesz basis for its linear span in $L^2(\mathbb{R})$. By the duality principle, the Gabor system $\mathcal{G}(h_n, \Lambda)$ is a frame for $L^2(\mathbb{R})$. ∎

5. Banach frames

In this section we extend the results about Gabor frame expansions in $L^2(\mathbb{R})$ and the sampling theorem in $\mathcal{F}_2^n(\mathbb{C})$ to a class of associated Banach spaces. This extension can be formulated in terms of Banach frames [23] and can be done conveniently with the theory of localized frames [26, 18].

5.1. **Banach frames.** The theory of localized frames asserts that every "nice" frame is automatically a Banach frame for an associated class of Banach spaces.

For the description of modulation spaces with Gabor frames we recall a precise statement from [26, Thm. 9].

Theorem 3. *Assume that $\Lambda \subseteq \mathbb{R}^2$ is a lattice, that $g \in M^1(\mathbb{R})$, and that $\{\pi_\lambda g : \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R})$.*

Then there exists a dual window $\gamma \in M^1(\mathbb{R})$, such that the corresponding frame expansion

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle \pi_\lambda \gamma \tag{5.1}$$

converges unconditionally in $M^p(\mathbb{R})$ for $1 \le p < \infty$ (and weak- in $M^\infty(\mathbb{R})$).*

A distribution f belongs to the modulation space $M^p(\mathbb{R})$, if and only if the frame coefficients $\langle f, \pi_\lambda g \rangle$ belong to $\ell^p(\Lambda)$. Furthermore, the following norm equivalence

holds on $M^p(\mathbb{R})$:

$$\|f\|_{M^p} \asymp \Big(\sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda g\rangle|^p\Big)^{1/p} \tag{5.2}$$

As a consequence of the duality theory (Theorem A) the dual window γ satisfies the biorthogonality condition $s(\Lambda)^{-1}\langle \gamma, \pi_\mu g\rangle = \delta_{\mu,0}$ for $\mu \in \Lambda^0$.

For $p = 2$ the properties (5.1) and (5.2) are the defining properties of a frame of a Hilbert space. By analogy for $p \neq 2$, we call a set satisfying (5.1) and (5.2) Banach frame for $M^p(\mathbb{R})$.

In this paper we have restricted ourselves to dimension $d = 1$ and the unweighted case, but the theory of localized frames offers much more general versions of Theorem 3.

5.2. **Explicit sampling formulas in $\mathcal{F}_p^n(\mathbb{C})$.** In this section we translate Theorem 3 into the language of polyanalytic functions and derive a sampling expansion, which in a sense is the dual of the polyanalytic interpolation formula (4.5) of Theorem 1.

Theorem 4. *Assume that $\Lambda \subseteq \mathbb{C}$ is a lattice and $s(\Lambda) < (n+1)^{-1}$.*

(i) Then F belongs to the true poly-Fock space $\mathcal{F}_p^n(\mathbb{C})$, if and only if the sequence with entries $e^{-\pi|\lambda|^2/2}F(\lambda)$ belongs to $\ell^p(\Lambda)$, with the norm equivalence

$$\|F\|_{\mathcal{F}_p^n} \asymp \Big(\sum_{\lambda\in\Lambda} |F(\lambda)|^p e^{-\pi p|\lambda|^2/2}\Big)^{1/p}.$$

(ii) Let

$$S_{\Lambda^0}^n(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} e^{\pi|z|^2}\left(\frac{d}{dz}\right)^n\left[e^{-\pi|z|^2}\frac{\sigma_{\Lambda^0}(z)^{n+1}}{n!z}\right] \tag{5.3}$$

be the interpolating function on the adjoint lattice Λ^0. Then every $F \in \mathcal{F}_p^{n+1}(\mathbb{C})$ can be written as

$$F(z) = \sum_{\lambda\in\Lambda} F(\lambda) e^{\pi\overline{\lambda}z-\pi|\lambda|^2} S_{\Lambda^0}^n(z-\lambda). \tag{5.4}$$

The sampling expansion converges in the norm of $\mathcal{F}_p^n(\mathbb{C})$ for $1 \le p < \infty$ and pointwise for $p = \infty$.

Proof. By Corollary 2 there exists an $f \in M^p(\mathbb{R})$, such that $F = \mathcal{B}^{n+1}f \in \mathcal{F}_p^n(\mathbb{C})$, more precisely, according to (2.7) $\mathcal{B}^{n+1}F(z) = e^{-i\pi x\xi}e^{-\pi|z|^2/2}\langle f, \pi_{(x,-\xi)}h_n\rangle$. Since $\pi_\lambda h_n$ is a Banach frame for $M^p(\mathbb{R})$ by Theorem 3, the norm equivalence

$$\|f\|_{M^p} \asymp \Big(\sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda h_n\rangle|^p\Big)^{1/p}$$

translates into the norm equivalence

$$\|F\|_{\mathcal{F}_p^n} \asymp \Big(\sum_{\lambda\in\Lambda} |F(\lambda)|^p e^{-\pi p|z|^2/2}\Big)^{1/p}, \qquad \forall F \in \mathcal{F}_p^n(\mathbb{C}).$$

We now apply the polyanalytic Bargmann transform to (5.1) and obtain a reconstruction formula for the samples of $F \in \mathcal{F}_p^{n+1}(\mathbb{C})$:

$$F(z) = \mathcal{B}^{n+1} f(z) = \sum_{\lambda \in \Lambda} F(\lambda) e^{-\frac{\pi}{2}|\lambda|^2} e^{-\pi i \lambda_1 \lambda_2} \mathcal{B}^{n+1} (\pi_\lambda \gamma)(z).$$

Now the intertwining property (4.1) gives

$$\begin{aligned} F(z) &= \sum_{\lambda \in \Lambda} F(\lambda) e^{-\frac{\pi}{2}|\lambda|^2} e^{-\pi i \lambda_1 \lambda_2} \beta_\lambda \mathcal{B}^{n+1} \gamma(z) \\ &= \sum_{\lambda \in \Lambda} F(\lambda) e^{-\pi|\lambda|^2} e^{\pi \overline{\lambda} z} \mathcal{B}^{n+1} \gamma(z - \lambda) . \end{aligned} \tag{5.5}$$

Since the frame expansion (5.1) converges in $M^p(\mathbb{C})$ and since $\mathcal{B}^{n+1}$ is an isometry from $M^p(\mathbb{R})$ onto $\mathcal{F}_p^n$, the sampling expansion (5.5) must converge in $\mathcal{F}_p^n(\mathbb{C})$.

The expansion (5.5) holds for every dual window $\gamma \in M^1(\mathbb{R})$. By choosing the particular window, we can derive a more explicit formula for $\mathcal{B}^{n+1}\gamma$. Since every dual window γ satisfies the biorthogonality relation $\delta_{\mu,0} = s(\Lambda)^{-1} \langle \gamma, \pi_\mu h_n \rangle = \mathcal{B}^{n+1}\gamma(\bar{\mu})\, e^{\pi i \mu_1 \mu_2 - \pi|\mu|^2/2}$, the true poly Bargmann transform of γ is an interpolating functions on Λ^0. Of all such functions we may therefore use the interpolating function $S_{\Lambda^0}^n = \mathcal{B}^{n+1}\gamma$ defined in (5.3) as the expanding function in (5.5). ∎

The following is a sampling theorem which can be applied to the vector valued situation studied in [28]. Again we emphasize that this is an explicit formula while the one obtained with the superframe representation is not, because we do not know the dual vectorial window explicitly.

Corollary 3. *If $s(\Lambda) < \frac{1}{n+1}$, then every $F \in \mathbf{F}_p^{n+1}(\mathbb{C})$ can be written as:*

$$F(z) = \sum_{\lambda \in \Lambda} F(\lambda) e^{\pi \overline{\lambda} z - \pi |\lambda|^2} \mathbf{S}_{\Lambda^0}^n (z - \lambda),$$

where

$$\mathbf{S}_{\Lambda^0}^n (z) = \sum_{k=0}^{n} S_{\Lambda^0}^k (z).$$

Proof. By Corollary 1, $\mathbf{F}_p^{n+1}(\mathbb{C})$ can be written as a direct sum of the spaces $\mathcal{F}_p^k(\mathbb{C})$. Thus, one can write $\mathbf{F} = \sum_{k=0}^n F_k$, with $F_k \in \mathcal{F}^{k+1}(\mathbb{C})$, and the result follows from Theorem 4. ∎

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